(A)

MAT 1322 A W2010 Wed. March 17th 8:30–9:50 Prof. Desjardins

MIDTERM TEST 2

Max = 20

Student Number: Solutions.

•Time: 80 min.

• Only basic scientific calculators are permitted (non-graphing, non-programmable, no integration or differentiation capabilities). Notes or books are not permitted.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite.

• The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.

(A)

1. (3 points) A detective finds a murder victim at 8am. At that time, the temperature of the body was 30°C. One hour later, it was 26°C. The body had been in a room with constant temperature 21°C. The victim had not been sick, so one can assume that at the time of the murder, the victim's body temperature was a normal 37.4°C. Use Newton's Law of Cooling to set up a differential equation for the temperature of the body, T. Solve the DE and estimate the time of the murder.

Newton's Law of Cooling $\frac{dT}{dt} = -k(T-21)$

separate variables $\frac{dT}{T-21} = -k\,dt$

integrate $\int \frac{dT}{T-21} = \int -k\,dt + C$

to get $\ln|T-21| = -kt + C$

exponentiate $T - 21 = Ae^{-kt}$

then the general solution is $T(t) = 21 + Ae^{-kt}$

$T(0) = 30 \Rightarrow 30 = 21 + A \Rightarrow A = 9$

So $T(t) = 21 + 9e^{-kt}$

$T(1) = 26 \Rightarrow 26 = 21 + 9e^{-k} \Rightarrow 9e^{-k} = 5$

So $k = -\ln(5/9) \approx 0.5878$

$\therefore$ $\boxed{T(t) = 21 + 9e^{-0.5878t}}$

murder occured when $T(t) = 37.4$

So $37.4 = 21 + 9e^{-0.5875t} \Rightarrow 9e^{-0.5878t} = 16.4$

$\therefore$ $t = \frac{\ln(16.4/9)}{-0.5878} \approx \boxed{-1.02 \text{ hr}}$

ie the murder occured at approx. $\boxed{7\text{am}}$

(A)

2. (3 points) Biologists place a herd of 1000 gnus on an island. They estimate that the carrying capacity of the island is 5000 gnus. Also, the relative growth rate in an unconstrained environment is estimated to be $k = 0.1$ per year. Assuming that the population follows the Logistic Model,
(i) write the differential equation that the population $P(t)$ will satisfy, where t is measured in years and
(ii) given that the solution of the Logistic equation is $P(t) = \dfrac{M}{1 + Ae^{-kt}}$, find the number of gnus on the island after 3 years.

i, $$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$ told $k = 0.1$, $M = 5000$

So $$\boxed{\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{5000}\right)}$$

ii, $$P(t) = \frac{M}{1+Ae^{-kt}} = \frac{5000}{1+Ae^{-0.1t}}$$

$$P(0) = 1000 \Rightarrow \frac{5000}{1+A} = 1000 \Rightarrow A = 4$$

So $$P(t) = \frac{5000}{1+4e^{-0.1t}}$$

then $$P(3) = \frac{5000}{1+4e^{-0.1(3)}} \approx \boxed{1262}$$

(A)

3. (4 points) Determine if the series converge or diverge. Explain your reasoning and demonstrate that any conditions required to use a particular test are satisfied.

(a) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-2n^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3+\cos(n)}{n}$

a) let $f(x) = x^2 e^{-2x^3}$

then $f(x) > 0$ for $x \geq 1$

and $f'(x) = 2x e^{-2x^3} - 6x^4 e^{-2x^3}$

$= 2x(1-3x^3)e^{-2x^3}$

< 0 for $x \geq 1$

use the Integral Test

$\int_1^{\infty} x^2 e^{-2x^3} dx$

$= \lim_{b\to\infty} \int_1^b x^2 e^{-2x^3} dx$

$= \lim_{b\to\infty} -\frac{1}{6} e^{-2x^3} \Big|_1^b$

$= \lim_{b\to\infty} -\frac{1}{6}\left(e^{-2b^3} - e^{-2}\right)$

$= \frac{e^{-2}}{6}$

$\therefore \int_1^{\infty} x^2 e^{-2x^3} dx$ converges

and so $\sum_{n=1}^{\infty} n^2 e^{-2n^3}$

converges

b) since $-1 \leq \cos(n) \leq 1$ for any n

$\frac{3+\cos(n)}{n} \geq \frac{2}{n}$

and so $\sum_{n=1}^{\infty} \frac{3+\cos(n)}{n} \geq \sum_{n=1}^{\infty} \frac{2}{n}$

and we know that

$\sum_{n=1}^{\infty} \frac{2}{n}$ diverges (p series)

$\therefore \sum_{n=1}^{\infty} \frac{3+\cos(n)}{n}$ diverges

by the Comparison Test

A

4. (4 points) Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(x+1)^n}{3^n(n+2)}$.

(centre $a=-1$)

use the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(x+1)^{n+1}}{3^{n+1}(n+3)}}{\frac{(x+1)^n}{3^n(n+2)}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \frac{3^n}{3^{n+1}} \frac{n+2}{n+3} \right|$$

$$= \lim_{n\to\infty} \frac{|x+1|}{3}\left(\frac{n+2}{n+3}\right)$$

$$= \frac{|x+1|}{3}$$

for convergence, need $\frac{|x+1|}{3} < 1$ or $|x+1| < 3$

so $\boxed{R=3}$

or $-4 < x < 2$

check endpoints: if $x=-4$, $\sum_{n=0}^{\infty} \frac{(x+1)^n}{3^n(n+2)} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n(n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$

which converges (AST)

if $x=2$, $\sum_{n=0}^{\infty} \frac{(x+1)^n}{3^n(n+2)} = \sum_{n=0}^{\infty} \frac{3^n}{3^n(n+2)} = \sum_{n=0}^{\infty} \frac{1}{n+2}$ which diverges (Limit Comparison)

$\therefore$ the interval of convergence is $\boxed{-4 \le x < 2}$

(A)

5. (4 points)
(i) Give the Maclaurin Series of $\cos(x^2)$ and deduce that of
$f(x) = \int_0^x \cos(t^2)\,dt$.
(ii) For $x = 1$, the result above gives an alternating series for $f(1) = \int_0^1 \cos(t^2)\,dt$. How many terms are required such that the error is at most 10^{-2}? What is this value for $f(1)$?

i) $$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \boxed{\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}}$$

$$f(x) = \int_0^x \cos(t^2)dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!} dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+1}}{(2n)!(4n+1)} \Big|_0^x$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!(4n+1)}}$$

ii) $$f(1) = \int_0^1 \cos(t^2)dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(4n+1)}$$

$$= 1 - \frac{1}{2!(5)} + \frac{1}{4!(9)} - \frac{1}{6!(13)} + \cdots$$

$$= 1 - \frac{1}{10} + \frac{1}{216} - \frac{1}{9360} + \cdots$$

notice that $\frac{1}{216} < 10^{-2}$, so only $\boxed{\text{first two terms}}$ are needed (alternating series)

and so $f(1) \simeq 1 - \frac{1}{10} = \boxed{0.90}$

Ⓐ

6. (2 points + 1 bonus point)
(i) Give an example of a convergent series that is not absolutely convergent.

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}$$ (Alternating Harmonic Series)

(ii) Give an example of a series where the general term, a_n, goes to 0 as $n \to \infty$, but the series is divergent.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$$ (Harmonic Series)

(iii) For what values of r will the series $\sum_{n=1}^{\infty} ar^{n-1}$ converge? What is the sum in that case?

it converges if $\boxed{|r| < 1}$

and the sum is $\boxed{\dfrac{a}{1-r}}$

(B)

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MIDTERM TEST 2

Max = 20

Student Number: $\boxed{\text{Solutions}}$

(See version A for more details)

•Time: 80 min.

• Only basic scientific calculators are permitted (non-graphing, non-programmable, no integration or differentiation capabilities). Notes or books are not permitted.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite.

• The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.

(B)

1. (3 points) A detective finds a murder victim at 8am. At that time, the temperature of the body was 28°C. One hour later, it was 24°C. The body had been in a room with constant temperature 21°C. The victim had not been sick, so one can assume that at the time of the murder, the victim's body temperature was a normal 37.4°C. Use Newton's Law of Cooling to set up a differential equation for the temperature of the body, T. Solve the DE and estimate the time of the murder.

(see Version A)

$$T(t) = 21 + Ae^{-kt}$$

$$T(0) = 28 \Rightarrow A = 7 \Rightarrow T(t) = 21 + 7e^{-kt}$$

$$T(1) = 24 \Rightarrow 24 = 21 + 7e^{-k}$$

so $7e^{-k} = 3 \Rightarrow k = -\ln(3/7) \approx 0.8473$

$$\therefore \boxed{T(t) = 21 + 7e^{-0.8473t}}$$

$$37.4 = 21 + 7e^{-0.8473t} \Rightarrow 7e^{-0.8473t} = 16.4$$

so $t = \dfrac{\ln(16.4/7)}{-0.8473} \approx \boxed{-1.00 \text{ hr}}$

$\therefore$ murder occured at approx. $\boxed{7\text{am}}$

(B)

2. (3 points) Biologists place a herd of 1000 gnus on an island. They estimate that the carrying capacity of the island is 4000 gnus. Also, the relative growth rate in an unconstrained environment is estimated to be $k = 0.1$ per year. Assuming that the population follows the Logistic Model,
(i) write the differential equation that the population $P(t)$ will satisfy, where t is measured in years and
(ii) given that the solution of the Logistic equation is $P(t) = \dfrac{M}{1 + Ae^{-kt}}$, find the number of gnus on the island after 2 years.

$k = 0.1$, $M = 4000$

i, $\boxed{\dfrac{dP}{dt} = 0.1P\left(1 - \dfrac{P}{4000}\right)}$

ii, $P(t) = \dfrac{4000}{1+Ae^{-0.1t}}$

$P(0) = 1000 \Rightarrow \dfrac{4000}{1+A} = 1000 \Rightarrow A = 3$

$P(t) = \dfrac{4000}{1+3e^{-0.1t}}$

then $P(2) = \dfrac{4000}{1+3e^{-0.1(2)}} \approx \boxed{1157}$

Ⓑ

3. (4 points) Determine if the series converge or diverge. Explain your reasoning and demonstrate that any conditions required to use a particular test are satisfied.

(a) $\sum_{n=1}^{\infty} 2n^2 e^{-n^3}$ (b) $\sum_{n=1}^{\infty} \frac{2-\sin(n)}{n}$

(See Version A)

a) $f(x) = 2x^2 e^{-x^3} > 0$ for $x \geq 1$

$f'(x) = 4xe^{-x^3} - 6x^4 e^{-x^3}$
$= 2x(2-3x^3)e^{-x^3}$
< 0 for $x \geq 1$

$\int_1^{\infty} 2x^2 e^{-x^3} dx$

$= \lim_{b\to\infty} \int_1^b 2x^2 e^{-x^3} dx$

$= \lim_{b\to\infty} -\frac{2}{3} e^{-x^3} \Big|_1^b$

$= \lim_{b\to\infty} -\frac{2}{3}\left(e^{-b^3} - e^{-1}\right)$

$= \frac{2e^{-1}}{3}$

$\therefore \int_1^{\infty} 2x^2 e^{-x^3} dx$ converges

So $\sum_{n=1}^{\infty} 2n^2 e^{-n^3}$ converges by Integral Test

b) since $-1 \leq \sin(n) \leq 1$ for all n

$\frac{2-\sin(n)}{n} \geq \frac{1}{n}$

So $\sum_{n=1}^{\infty} \frac{2-\sin(n)}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges

$\therefore \sum_{n=1}^{\infty} \frac{2-\sin(n)}{n}$ diverges by Comparison

(B)

4. (4 points) Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(x+2)^n}{4^n(n+3)}$.

(See version A) (Centre $a=-2$)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x+2|}{4} \left(\frac{n+3}{n+4}\right) = \frac{|x+2|}{4}$$

need $|x+2| < 4$ so $\boxed{R=4}$

or $-6 < x < 2$

if $x=-6$, $\sum_{n=0}^{\infty} \frac{(x+2)^n}{4^n(n+3)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$ converges (AST)

if $x=2$, $\sum_{n=0}^{\infty} \frac{(x+2)^n}{4^n(n+3)} = \sum_{n=0}^{\infty} \frac{1}{n+3}$ diverges

so interval is $\boxed{-6 \le x < 2}$

B

5. (4 points)
(i) Give the Maclaurin Series of $\sin(x^2)$ and deduce that of $f(x) = \int_0^x \sin(t^2)\,dt$.

(ii) For $x = 1$, the result above gives an alternating series for $f(1) = \int_0^1 \sin(t^2)\,dt$. How many terms are required such that the error is at most 10^{-2}? What is this value for $f(1)$?

i) $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \boxed{\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}}$

then $f(x) = \int_0^x \sin(t^2)\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!}\,dt$

$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)}}$

ii) $f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(4n+3)}$

$= \frac{1}{3} - \frac{1}{3!\,(7)} + \frac{1}{5!\,(11)} - \frac{1}{7!\,(15)} + \cdots$

$= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \cdots$

notice that $\frac{1}{1320} < 10^{-2}$ so need only

$\boxed{\text{2 terms}}$

and then $f(1) \approx \frac{1}{3} - \frac{1}{42} = \boxed{0.31}$

B

6. (2 points + 1 bonus point)
(i) Give an example of a series where the general term, a_n, goes to 0 as $n \to \infty$, but the series is divergent.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$$

(ii) For what values of r will the series $\sum_{n=1}^{\infty} ar^{n-1}$ converge? What is the sum in that case?

$\boxed{|r|<1}$ $\boxed{\dfrac{a}{1-r}}$

(iii) Give an example of a convergent series that is not absolutely convergent.

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}$$